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Classification of Solutions in Topologically Massive Gravity

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Abstract

We study exact solutions of three-dimensional gravity with a cosmological constant and a gravitational Chern–Simons term: the theory known as topologically massive gravity. After reviewing the algebraic classification, we show that if a solution has curvature of algebraic type D, then it is biaxially squashed AdS_3. Applying the classification, we provide a comprehensive review of the literature, showing that most known solutions are locally equivalent to biaxially squashed AdS_3 or to AdS pp-waves.

Contents

1 Introduction

Three-dimensional gravity with a gravitational Chern–Simons term, and possibly with a cosmological constant, is known as topologically massive gravity (TMG) [1, 2, 3], and is described by the action

$$I = \frac{1}{16\pi G}\int \mathrm{d}^3x\,\sqrt{-g}\left[R - 2\Lambda + \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\rho}\right)\right], \qquad (1.1)$$

where G is Newton's constant, Λ is the cosmological constant and μ is a mass parameter. The model supports a black hole solution [4, 5] for $\Lambda < 0$, and for a generic value of the Chern–Simons coupling constant μ it has a propagating massive graviton. The theory also admits a dual boundary conformal field theory description. All these properties make the model a natural one in which to study various aspects of non-perturbative gravity that seem to be forbiddingly difficult in higher dimensions.

In Einstein gravity, for which the Chern–Simons term is absent, and with a negative cosmological constant, there have been attempts to solve the theory exactly [6, 7, 8], by using knowledge of the solution space for fixed boundary conditions. Certain problems encountered in giving a physical interpretation to the partition function [8] might be circumvented by considering TMG instead, and, in particular, at a critical value of the Chern–Simons coupling constant: $\mu = \sqrt{-\Lambda}$ [9]. At this chiral point, with the further assumptions that $G > 0$, and that the standard Brown–Henneaux boundary conditions [10] hold, the theory is known as chiral gravity. This proposal motivates an exhaustive investigation of the solution space of TMG, in which the Chern–Simons coupling plays a non-trivial rôle.

The field equation for TMG, obtained by varying (1.1) with respect to the metric, is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0\,, \qquad (1.2)$$

where $C_{\mu\nu}$ is the Cotton tensor, which is a symmetric and traceless tensor defined as

$$C_{\mu\nu} = \epsilon_\mu{}^{\rho\sigma}\nabla_\rho(R_{\sigma\nu} - \tfrac{1}{4}Rg_{\sigma\nu})\,. \qquad (1.3)$$

$\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor, related by $\epsilon_{\mu\nu\rho} = \sqrt{-g}\varepsilon_{\mu\nu\rho}$ to the weight $+1$ tensor density $\varepsilon_{\mu\nu\rho}$. Strictly speaking, the Cotton and Levi-Civita tensors are pseudotensors — they change sign under a parity transformation — and so the field equation is not quite tensorial. When specifying any solution, it is therefore essential to state both the metric and the convention for $\epsilon_{\mu\nu\rho}$; we take $\varepsilon_{012} = +1$. We usually consider the cosmological constant to be negative:

$\Lambda = -m^2$, where $|m|$ is the inverse AdS radius. However, we shall sometimes consider a vanishing cosmological constant or, by taking $m \to \mathrm{i}m$, a positive cosmological constant $\Lambda = m^2$. From the trace of the field equation, we deduce that $R = 6\Lambda$, and so the field equation can be written more simply as

$$R_{\mu\nu} - 2\Lambda g_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\rho\sigma}\nabla_\rho R_{\sigma\nu} = 0\,. \tag{1.4}$$

Henceforth, it suffices to consider only this form of the field equation.

We shall only consider local solutions of TMG. In three dimensions, the Riemann and Ricci tensors both have 6 independent components; these tensors are related by

$$R_{\mu\nu\rho\sigma} = 2R_{\mu[\rho}g_{\sigma]\nu} - 2R_{\nu[\rho}g_{\sigma]\mu} - Rg_{\mu[\rho}g_{\sigma]\nu}\,. \tag{1.5}$$

Any Einstein metric is therefore maximally symmetric, i.e. locally flat for $\Lambda = 0$, anti-de Sitter for $\Lambda < 0$, or de Sitter for $\Lambda > 0$. These are automatically solutions of TMG, since they have a vanishing Cotton tensor; in fact, a solution of TMG has a vanishing Cotton tensor if and only if it is Einstein with $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$.

We henceforth consider non-trivial exact solutions of TMG that have a non-vanishing Cotton tensor, which are not Einstein, but it turns out that remarkably few are known. Although there is much literature about exact solutions of TMG, almost all local metrics reduce to three particular solutions:

1. Timelike-squashed AdS_3 [11, 12], which is a biaxially squashed AdS_3 and can be regarded as a timelike fibration over $\mathbb{H}^2$.

2. Spacelike-squashed AdS_3 [13], which is a biaxially squashed AdS_3 and can be regarded as a spacelike fibration over AdS_2.

3. AdS pp-waves [14, 15, 16], which are generalizations of pp-waves to include a cosmological constant.

The only solutions in the literature that are not locally (special cases of) timelike- or spacelike-squashed AdS_3 or an AdS pp-wave are the general triaxially squashed AdS_3 solutions of Nutku and Baekler [17], given in their equations (4.1), (4.6) and (4.8), also found by Ortiz [18], given in his equation (5.4). These exceptional triaxially squashed solutions solve TMG without a cosmological constant, generalizing the timelike- and spacelike-squashed AdS_3 solutions, which have only biaxial squashing.

In this paper, we only consider local solutions, and do not distinguish between different global identifications. The global solutions of the theory without the Chern–Simons term are well-known; they are obtainable by making conformal transformations of the AdS_3 metric that preserve the Brown–Henneaux asymptotics (see, for example, [19] for a review). These include the BTZ black hole [4, 5], which is obtained from discrete quotienting of AdS_3. Similarly, discrete quotients of the squashed AdS_3 solutions of TMG give black holes [11, 20, 12, 21, 13]. Recently, these black holes were systematically studied in [22][1].

As in four-dimensional general relativity, one of the problems when surveying the literature of exact solutions is to disentangle genuinely new solutions from those that are already known but written in different coordinate systems. One approach to bringing some order to the compilation is to use the algebraic classification of curvature. We can classify the Cotton tensor [23, 24, 25] (see also [26, 27]), analogous to the four-dimensional Petrov classification of the Weyl tensor. Alternatively, we can classify the three-dimensional traceless Ricci tensor [28, 24, 25] (see also [27]), analogous to the Segre classification of the energy-momentum tensor in four-dimensional general relativity. In fact, a consequence of the field equation is that the traceless Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}$ and the Cotton tensor are proportional:

$$S_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0\,. \tag{1.6}$$

Thus the Petrov classification of the Cotton tensor and the Segre classification of the traceless Ricci tensor are equivalent in TMG; we shall refer to it as the "Petrov–Segre classification". In this paper, we survey all known solutions of TMG, determining their Petrov–Segre types and equivalences under local coordinate transformations, and study to what degree an exact solution can be determined by specifying its Killing symmetries and its Petrov–Segre type.

The problem of finding all solutions of TMG with a given set of Killing symmetries and with a specified Petrov–Segre class is in general highly non-trivial. A noteworthy exception is that by requiring the existence of a null Killing vector k, the solutions to TMG can be shown to be of Petrov–Segre type N, with the traceless Ricci tensor $S_{\mu\nu}$ proportional to $k_\mu\, k_\nu$ [29]. These solutions are known as AdS pp-waves [14], and they have been rediscovered on a number of occasions.

Here we shall also show that a Petrov–Segre type D solution of TMG must be a biaxially squashed AdS_3. Our main application of this result is that it provides an algorithm for

[1] Because the squashing parameters are constants, we call them squashed AdS_3 black holes, rather than "warped" AdS_3 black holes as referred to by [22].

testing whether a solution is a biaxially squashed AdS_3 in some coordinate system. The power of this result is illustrated in Appendices A.1 and A.2, where we show that various solutions in the literature are, by coordinate transformation, biaxially squashed AdS_3 in disguise, despite a smörgåsbord of coordinate systems to contend with.

The outline of the rest of this paper is as follows. In Section 2, we review the Petrov–Segre algebraic classification of curvature in three dimensions. In Section 3, we prove that type D spacetimes that are solutions of TMG must be squashed AdS_3. We complete the enumeration of the known solutions of TMG with a review of the pp-wave solution in Section 4. After presenting our conclusions in Section 5, we give, in the Appendix, a comprehensive review of the literature of previously known TMG solutions, showing in particular that most are related by local coordinate transformations to the squashed AdS_3 and AdS pp-wave solutions.

2 Algebraic classification of curvature

A considerable effort has gone into finding exact solutions of four-dimensional general relativity [30]. There are four main classification schemes: symmetry groups, such as isometry groups; algebraic classification of the Weyl tensor (Petrov classification); algebraic classification of the traceless Ricci tensor (Segre classification); and solutions admitting special vector or tensor fields that satisfy certain geometrically meaningful differential constraints. These ideas motivate approaches to finding and classifying exact solutions of TMG. A difference, specific to TMG, is that the three-dimensional analogues of algebraically classifying curvature, which are to classify the Cotton tensor and the traceless Ricci tensor, coincide in view of (1.6). In this paper, we focus on algebraic classification and, to a lesser extent, symmetry; in a subsequent paper [31], the focus is on solutions with special null vector fields.

2.1 Comparison of three and four dimensions

In four-dimensional general relativity, the 20 independent components of the Riemann tensor split into: 10 independent components of the Weyl tensor $C_{\mu\nu\rho\sigma}$; 9 independent components of the traceless Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$; and 1 component of the Ricci scalar R. There are two independent algebraic classifications of curvature: the Petrov classification of the Weyl tensor (see, for example, Chapter 4 of [30]), and the Segre (or Plebański) classification of the traceless Ricci tensor (see, for example, Chapter 5 of [30]). There are

several formulations, but one is that they classify according to the eigenvalues of linear maps associated with the curvature tensors, in particular the algebraic and geometric multiplicities of the eigenvalues.

In four dimensions, the 10 real components of the Weyl tensor can be packaged into a traceless and symmetric 3×3 complex matrix C_{ab}. One formulation of the four-dimensional Petrov classification is algebraic classification of $C^a{}_b$, regarded as a linear map between complex 3-vectors. However, in three dimensions the Riemann tensor can be expressed in terms of the Ricci tensor, so the Weyl tensor vanishes and there is no direct analogue of the Petrov classification. Instead, the rôle of the conformal tensor in three dimensions is played by the Cotton tensor; it is analogous to the Weyl tensor in four dimensions and higher, in that its vanishing is equivalent to conformal flatness. A three-dimensional analogue of the Petrov classification is algebraic classification of the Cotton tensor $C^a{}_b$, regarded as a linear map between 3-vectors. The three-dimensional Petrov classification of the Cotton tensor has been studied in [23, 24, 25, 26, 27].

In four dimensions, the Segre classification is algebraic classification of the traceless Ricci tensor $S^a{}_b = R^a{}_b - \frac{1}{4}R\delta^a_b$, regarded as a linear map between 4-vectors. By the Einstein equation, this is equivalent to algebraic classification of the energy-momentum tensor. There is a natural three-dimensional analogue: algebraic classification of $S^a{}_b = R^a{}_b - \frac{1}{3}R\delta^a_b$, regarded as a linear map between 3-vectors. The three-dimensional Segre classification of the traceless Ricci tensor has been studied in [28, 24, 25, 27].

2.2 Petrov–Segre classification in TMG

For a general three-dimensional theory, these analogues of the Petrov and Segre classifications are distinct. However, in TMG the Cotton tensor and the traceless Ricci tensor are proportional (1.6), so the two classifications coincide; in this context, we call it the Petrov–Segre classification. It might be better to regard it as classification of the traceless Ricci tensor, since the Cotton tensor comes from differentiation of the Ricci tensor and so is less fundamental, however the classification is algebraic.

The Petrov–Segre classification is given in Table 1. (l^a, m^a, n^a) represents a null frame: l and n are null vectors, m is a unit spacelike vector, and their only non-vanishing inner products are $l^a n_a = -1$ and $m^a m_a = 1$, so $\eta_{ab} = -l_a n_b - n_a l_b + m_a m_b$ is flat. We can also define the unit timelike vector $t^a = (l^a + n^a)/\sqrt{2}$ and unit spacelike vector $z^a = (l^a - n^a)/\sqrt{2}$. $\alpha = \alpha(x^\mu)$ and $\beta = \beta(x^\mu)$ are real functions, β is not identically zero, and $\beta \neq \pm 3\alpha$ for type $\mathrm{I}_\mathbb{R}$. It is possible to choose $\lambda = \pm 1$ and $\tau = \pm 1$.

Petrov	Segre	Canonical S_{ab}	$S^a{}_b$ Jordan form	Jordan basis	Minimal polynomial
O	$[(11,1)]$	0	0	(l^b, n^b, m^b)	S
N	$[(12)]$	$\lambda l_a l_b$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(l^b, -\lambda n^b, m^b)$	S^2
$\mathrm{D_t}$	$[(11),1]$	$\alpha(\eta_{ab} + 3t_a t_b)$	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$	(m^b, z^b, t^b)	$(\mathsf{S} + 2\alpha\mathsf{I})(\mathsf{S} - \alpha\mathsf{I})$
$\mathrm{D_s}$	$[1(1,1)]$	$\alpha(\eta_{ab} - 3m_a m_b)$	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$	(t^b, z^b, m^b)	$(\mathsf{S} + 2\alpha\mathsf{I})(\mathsf{S} - \alpha\mathsf{I})$
III	$[3]$	$2\tau l_{(a} m_{b)}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$(l^b, \tau m^b, -n^b)$	S^3
II	$[12]$	$\alpha(\eta_{ab} - 3m_a m_b)$ $+\lambda l_a l_b$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$	$(l^b, -\lambda n^b, m^b)$	$(\mathsf{S} + 2\alpha\mathsf{I})(\mathsf{S} - \alpha\mathsf{I})^2$
$\mathrm{I}_{\mathbb{R}}$	$[11,1]$	$\alpha(\eta_{ab} - 3m_a m_b)$ $-\beta\,(l_a l_b + n_a n_b)$	$\begin{pmatrix} \alpha+\beta & 0 & 0 \\ 0 & \alpha-\beta & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$	$(l^b + n^b, l^b - n^b, m^b)$	$(\mathsf{S} + 2\alpha\mathsf{I})(\mathsf{S} - \alpha\mathsf{I} + \beta\mathsf{I})(\mathsf{S} - \alpha\mathsf{I} - \beta\mathsf{I})$
$\mathrm{I}_{\mathbb{C}}$	$[1z\bar{z}]$	$\alpha(\eta_{ab} - 3m_a m_b)$ $-\beta\,(l_a l_b - n_a n_b)$	$\begin{pmatrix} \alpha+\mathrm{i}\beta & 0 & 0 \\ 0 & \alpha-\mathrm{i}\beta & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$	$(l^b + \mathrm{i}n^b, l^b - \mathrm{i}n^b, m^b)$	$(\mathsf{S} + 2\alpha\mathsf{I})(\mathsf{S} - \alpha\mathsf{I} + \mathrm{i}\beta\mathsf{I})(\mathsf{S} - \alpha\mathsf{I} - \mathrm{i}\beta\mathsf{I})$

Table 1: Petrov–Segre classification. This shows, for each Petrov–Segre type, the canonical form of the traceless Ricci tensor, its Jordan normal form, and the monic polynomial of $S^a{}_b$ of minimal degree that vanishes. Here, α and β are functions of spacetime, λ and τ are constants; l and n are null, m and z are spacelike, t is timelike; the matrix S has entries $S^a{}_b$.

The first and second columns of the table respectively name the types by analogy with the four-dimensional Petrov classification of the Weyl tensor and the Segre classification of the traceless Ricci tensor. The third column gives a canonical form of the traceless Ricci tensor in terms of the null frame. The fourth column presents the traceless Ricci tensor in Jordan normal form.[2] The fifth column provides the basis for Jordan normal form. The final column gives the minimal polynomial, which is the unique monic polynomial of minimum degree satisfied by $S^a{}_b$.

To precisely determine the Petrov–Segre type, we can find the Jordan normal form of $S^a{}_b$. This is equivalent to finding the minimal polynomial of $S^a{}_b$. In our three-dimensional situation, this is also equivalent to finding the eigenvalues of $S^a{}_b$, along with their algebraic and geometric multiplicities. The Jordan normal form of $S^a{}_b$ is a convenient way of encoding any of this information. In Segre notation, the symbols 1, 2, 3 denote sizes of Jordan blocks. Round brackets group together Jordan blocks that correspond to the same eigenvalue. Where there is a comma, Jordan blocks before the comma correspond to spacelike eigenvectors, and blocks after correspond to timelike eigenvectors. This refines the Petrov classification so that type D spacetimes are split according to whether the one-dimensional eigenspace of $S^\mu{}_\nu$ is timelike, which we denote as type $\mathrm{D_t}$, or spacelike, which we denote as type $\mathrm{D_s}$. Jordan normal form by itself does not distinguish between types $\mathrm{D_t}$ and $\mathrm{D_s}$. Although $S_{\mu\nu}$ is symmetric, $S^\mu{}_\nu$ is not symmetric and can have complex eigenvalues. We can refine the Petrov classification by splitting Petrov type I spacetimes into type $\mathrm{I}_\mathbb{R}$ (or $[11,1]$ in Segre notation), which has three real distinct eigenvalues, and type $\mathrm{I}_\mathbb{C}$ (or $[1z\bar{z}]$ in Segre notation), which has one real and two complex conjugate eigenvalues. Each Jordan normal form has its own minimal polynomial. Indeed, a standard method of obtaining the Jordan normal form is to find the eigenvalues from the characteristic equation, and then to find the minimal polynomial, instead of computing the eigenvectors.

[2] Any square matrix can be transformed, by a similarity transformation, to Jordan normal form, which is block diagonal:

$$\mathsf{M} = \begin{pmatrix} \mathsf{J}_1 & & \\ & \ddots & \\ & & \mathsf{J}_p \end{pmatrix},$$

where each Jordan block J_i is of the form

$$\mathsf{J}_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix},$$

i.e. λ_i times an identity matrix plus 1's on the superdiagonal.

Note that determining the eigenvalues of $S^a{}_b$ and their algebraic multiplicities is equivalent to finding the scalar invariants[3]

$$I := S^\mu{}_\nu S^\nu{}_\mu = \mathrm{tr}(\mathsf{S}^2)\,, \quad J := S^\mu{}_\nu S^\nu{}_\rho S^\rho{}_\mu = \mathrm{tr}(\mathsf{S}^3)\,. \tag{2.1}$$

Petrov–Segre types O, N and III satisfy the syzygies $I = J = 0$, types $\mathrm{D_t}$, $\mathrm{D_s}$ and II satisfy the syzygy $I^3 = 6J^2 \neq 0$, and the most general types $\mathrm{I_{\mathbb{R}}}$ and $\mathrm{I_{\mathbb{C}}}$ have respectively $I^3 > 6J^2$ and $I^3 < 6J^2$.

We record here the Petrov–Segre types of the previously known solutions of TMG: $\mathrm{AdS_3}$ is type O, AdS pp-waves are type N, timelike-squashed $\mathrm{AdS_3}$ is type $\mathrm{D_t}$, spacelike-squashed $\mathrm{AdS_3}$ is type $\mathrm{D_s}$, and triaxially squashed $\mathrm{AdS_3}$ is type $\mathrm{I_{\mathbb{R}}}$. There are no solutions of types III, II and $\mathrm{I_{\mathbb{C}}}$ in the literature. Examples of type III and type II solutions are found in [31].

3 Squashed $\mathrm{AdS_3}$

We now prove that a Petrov–Segre type D solution of TMG must be biaxially squashed $\mathrm{AdS_3}$. First, we discuss properties of the squashed $\mathrm{AdS_3}$ solutions.

3.1 Biaxially squashed $\mathrm{AdS_3}$

$\mathrm{AdS_3}$ is a homogeneous spacetime that is in fact isomorphic to the group manifold $\mathrm{SU}(1,1)$ (or, locally, $\mathrm{SO}(1,2)$). It can be obtained from S^3 by analytic continuation. If we begin with the usual left-invariant 1-forms of SU(2), defined in terms of Euler angles by

$$\sigma_1 = \cos\psi\,\mathrm{d}\theta + \sin\psi\,\sin\theta\,\mathrm{d}\phi\,, \quad \sigma_2 = -\sin\psi\,\mathrm{d}\theta + \cos\psi\,\sin\theta\,\mathrm{d}\phi\,, \quad \sigma_3 = \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\,, \tag{3.1}$$

we can consider "triaxially squashed" left-invariant metrics on S^3,

$$\mathrm{d}s^2 = \frac{1}{4}\sum_{i=1}^{3} \lambda_i^2\,\sigma_i^2\,. \tag{3.2}$$

The round S^3, which is bi-invariant, is obtained by setting $\lambda_1 = \lambda_2 = \lambda_3$. If the squashing parameters λ_i are all taken to be equal to 1, then this gives the unit-radius S^3. Taking $\lambda_1 = \lambda_2$, and $\lambda_3 = \lambda$, and for convenience setting the scale by choosing $\lambda_1 = \lambda_2 = 1$, we

[3] The normalisations $I = \frac{1}{2}S^\mu{}_\nu S^\nu{}_\mu$ and $J = \frac{1}{6}S^\mu{}_\nu S^\nu{}_\rho S^\rho{}_\mu$ are also frequently used in the analogous four-dimensional literature.

obtain the 1-parameter family of biaxially squashed S^3 metrics

$$ds^2 = \tfrac{1}{4}\lambda^2(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 + \tfrac{1}{4}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{3.3}$$

which are sometimes known as Berger metrics.

If we now make the analytic continuation and redefinition

$$\theta \longrightarrow \mathrm{i}\,\theta\,, \qquad \psi = \tau\,, \tag{3.4}$$

and then reverse the sign of the metric (thus giving, in the "round" case, a negative cosmological constant), we obtain the 1-parameter family of biaxial timelike-squashed AdS_3 metrics

$$ds^2 = -\tfrac{1}{4}\lambda^2(\mathrm{d}\tau + \cosh\theta\,\mathrm{d}\phi)^2 + \tfrac{1}{4}(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,. \tag{3.5}$$

We refer to this as a timelike squashing, since it is a timelike fibre that is scaled relative to the size of the 2-dimensional base space (the hyperbolic plane).

There is a second type of biaxial squashing of AdS_3, which we refer to as a spacelike squashing. This is obtained by again starting from (3.3), but now making the analytic continuations and redefinitions

$$\theta = \tfrac{1}{2}\pi - \mathrm{i}\rho\,, \qquad \phi = \tau\,, \qquad \psi = \mathrm{i}z\,. \tag{3.6}$$

This gives, after reversing the sign of the original biaxially squashed S^3 metric (3.3), the 1-parameter family of biaxial spacelike-squashed AdS_3 metrics

$$ds^2 = -\tfrac{1}{4}\cosh^2\rho\,\mathrm{d}\tau^2 + \tfrac{1}{4}\mathrm{d}\rho^2 + \tfrac{1}{4}\lambda^2\,(\mathrm{d}z + \sinh\rho\,\mathrm{d}\tau)^2\,. \tag{3.7}$$

Here, the 2-dimensional base spacetime is AdS_2.

The timelike-squashed AdS_3 solution of TMG (with $\Lambda = -m^2$) has the metric

$$ds^2 = \frac{9}{\mu^2 + 27m^2}\Big[- \lambda^2(\mathrm{d}\tau + \cosh\theta\,\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2\Big]\,, \tag{3.8}$$

where the squashing parameter is defined as

$$\lambda^2 = \frac{4\mu^2}{\mu^2 + 27m^2} \tag{3.9}$$

If $\mu = \pm 3m$, then the squashing parameter becomes 1, and the metric is the standard

"round" AdS_3. The traceless Ricci tensor is

$$S_{\mu\nu} = (\tfrac{1}{9}\mu^2 - m^2)(g_{\mu\nu} + 3k_\mu k_\nu)\,, \tag{3.10}$$

where k is the unit timelike Killing vector

$$k^\mu \partial_\mu = \frac{\mu^2 + 27m^2}{6\mu} \frac{\partial}{\partial \tau}\,, \tag{3.11}$$

which has, by lowering the index, the associated 1-form

$$k_\mu \, \mathrm{d}x^\mu = -\frac{6\mu}{\mu^2 + 27m^2}(\mathrm{d}\tau + \cosh\theta \, \mathrm{d}\phi)\,. \tag{3.12}$$

Therefore the metric has Petrov–Segre type $\mathrm{D_t}$. Examples of timelike-squashed AdS_3 appearing in the literature are given in Appendix A.1.

The spacelike-squashed AdS_3 solution of TMG has the metric

$$\mathrm{d}s^2 = \frac{9}{\mu^2 + 27m^2}\Big[-\cosh^2\rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \lambda^2(\mathrm{d}z + \sinh\rho \, \mathrm{d}\tau)^2\Big]\,, \tag{3.13}$$

where again the squashing parameter is given by (3.9). Again, if $\mu = \pm 3m$, then the squashing parameter becomes 1 and we have round AdS_3. The traceless Ricci tensor is

$$S_{\mu\nu} = (\tfrac{1}{9}\mu^2 - m^2)(g_{\mu\nu} - 3k_\mu k_\nu)\,, \tag{3.14}$$

where k is the unit spacelike Killing vector

$$k^\mu \partial_\mu = \frac{\mu^2 + 27m^2}{6\mu} \frac{\partial}{\partial z}\,, \tag{3.15}$$

which has the associated 1-form

$$k_\mu \, \mathrm{d}x^\mu = \frac{6\mu}{\mu^2 + 27m^2}(\mathrm{d}z + \sinh\rho \, \mathrm{d}\tau)\,. \tag{3.16}$$

Therefore the metric has Petrov–Segre type $\mathrm{D_s}$. Examples of spacelike-squashed AdS_3 appearing in the literature are given in Appendix A.2.

For both the timelike- and spacelike-squashed AdS_3 solutions, the covariant derivative of the Killing vector k is

$$\nabla_\mu k_\nu = \tfrac{1}{3}\mu \epsilon_{\mu\nu\rho} k^\rho\,. \tag{3.17}$$

In combination with the expressions for the traceless Ricci tensor in terms of k — (3.10) and (3.14) — it is straightforward to check that the field equation (1.4) is solved.

We also have solutions for a positive cosmological constant, by taking $m \to im$. Although there might appear to be a problem for $\Lambda = \frac{1}{27}\mu^2$, for which the 2-dimensional base is flat, this is merely an artifact of a bad coordinate choice. For $\Lambda > \frac{1}{27}\mu^2$, we have squashing of de Sitter spacetime.

3.2 From type D to squashed AdS$_3$

Many of the solutions of TMG in the literature turn out locally to be biaxially squashed AdS$_3$. In light of this, it is useful to obtain a result that enables us to determine whether or not a given solution is biaxially squashed AdS$_3$, regardless of what coordinate system it is presented to us.

Our main result in this section will be to show that a Petrov–Segre type D solution of TMG is biaxially squashed AdS$_3$. More specifically: a type D$_\mathrm{t}$ solution, which has a traceless Ricci tensor of the form $S_{\mu\nu} = \alpha(g_{\mu\nu} + 3k_\mu k_\nu)$ with $k^\mu k_\mu = -1$ and $\alpha = \alpha(x^\mu)$ a scalar function, is timelike-squashed AdS$_3$; a type D$_\mathrm{s}$ solution, which has $S_{\mu\nu} = \alpha(g_{\mu\nu} - 3k_\mu k_\nu)$ with $k^\mu k_\mu = 1$, is spacelike-squashed AdS$_3$. We shall make make use of this result in Appendices A.1 and A.2, where we show that various solutions in the literature are, by coordinate transformation, biaxially squashed AdS$_3$ in disguise. Analogously, all four-dimensional type D spacetimes that are vacuum are known [32], as are those that admit a cosmological constant [33, 34].

We begin by noting that Petrov–Segre type D$_\mathrm{t}$ solution has a Ricci tensor of the form

$$R_{\mu\nu} = (p - 2m^2)g_{\mu\nu} + 3pk_\mu k_\nu\,, \tag{3.18}$$

where k is a timelike vector field normalized so that $k^\mu k_\mu = -1$, and p is a scalar function that is not identically zero. Our first aim is to show that p is constant and that $\nabla_\mu k_\nu = \frac{1}{3}\epsilon_{\mu\nu\rho}k^\rho$. We then perform a dimensional reduction on the Killing vector k, showing that the 2-dimensional base space is Einstein, and then deducing that the spacetime is squashed AdS$_3$.

Using the vector field k, we perform a $2+1$ split of the field equation components. We define $h^\mu{}_\nu = \delta^\mu_\nu + k^\mu k_\nu$, which satisfies $h^\mu{}_\nu k^\nu = 0$, to project out tensor components orthogonal to k. Contracting the field equation (1.4) with $k^\mu k^\nu$, $k^\mu h^\nu{}_\sigma$, $h^\mu{}_\rho k^\nu$ and $h^\mu{}_\rho h^\nu{}_\sigma$

gives a decomposition into four simpler equations. We hence obtain

$$\epsilon^{\mu\nu\rho}k_\mu\partial_\nu k_\rho = \tfrac{2}{3}\mu\,, \tag{3.19}$$

$$\epsilon^{\mu\nu\rho}k_\nu\partial_\rho p = 0\,, \tag{3.20}$$

$$p h^\sigma{}_\rho \epsilon^{\mu\nu\rho}\partial_\mu k_\nu = \tfrac{2}{3}\epsilon^{\sigma\mu\nu}k_\mu\partial_\nu p\,, \tag{3.21}$$

$$p h_{\mu\nu} = \mu^{-1}(3p\epsilon_\mu{}^{\rho\sigma}k_\rho\nabla_\sigma k_\nu + h_{\mu\rho}h_{\nu\sigma}\epsilon^{\rho\sigma\tau}\partial_\tau p)\,. \tag{3.22}$$

(Minus) the quantity on the left hand side of (3.19) is known as the scalar twist of k, and is a constant here. Substitution of (3.20) into (3.21) shows that $\epsilon^{\mu\nu\rho}\partial_\mu k_\nu$ is proportional to k^ρ; the normalization is fixed by (3.19). We hence obtain

$$\partial_{[\mu}k_{\nu]} = \tfrac{1}{3}\mu\epsilon_{\mu\nu\rho}k^\rho\,, \tag{3.23}$$

and then (3.19) is redundant. Substituting (3.20) into (3.22), we have

$$p h_{\mu\nu} = \mu^{-1}(3p\epsilon_\mu{}^{\rho\sigma}k_\rho\nabla_\sigma k_\nu + \epsilon_{\mu\nu\rho}\partial^\rho p)\,. \tag{3.24}$$

We now show, by contradiction, that p must be constant. If p is not constant, then equation (3.20) implies that $k_\mu = q\partial_\mu p$ for some scalar function q, i.e. k is hypersurface-orthogonal. However, because $\mu \neq 0$, we would have a contradiction with (3.19). Therefore p is constant. Thus the entire content of the field equation reduces to p being constant, (3.23), and, from (3.24),

$$h_{\mu\nu} = 3\mu^{-1}\epsilon_\mu{}^{\rho\sigma}k_\rho\nabla_\sigma k_\nu\,. \tag{3.25}$$

Our next step is to strengthen (3.23) to (3.17). The Bianchi identity $\nabla_{[\mu}R_{\nu\rho]\sigma\tau} = 0$ is equivalent in three dimensions to the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, so in TMG is just $\nabla^\mu R_{\mu\nu} = 0$, which then gives

$$\nabla_\mu k^\mu = 0\,, \tag{3.26}$$

$$k^\mu\nabla_\mu k^\nu = 0\,. \tag{3.27}$$

We now extend k to an orthonormal frame, choosing the components of k^a to be $(k^0, k^1, k^2) = (1, 0, 0)$. Defining $K_{\mu\nu} := \nabla_\mu k_\nu$, from $\nabla_\mu(k^\nu k_\nu) = 0$ we have $K_{00} = K_{10} = K_{20} = 0$, and from the Bianchi identity we further have $K_{11} + K_{22} = 0$ and $K_{01} = K_{02} = 0$. Then, from

(3.25), we have $1 = h_{11} = -3\mu^{-1}K_{21}$, $1 = h_{22} = 3\mu^{-1}K_{12}$, and $0 = h_{12} = -3\mu^{-1}K_{22}$. The upshot is that the only non-vanishing components of K_{ab} are $K_{12} = -K_{21} = \frac{1}{3}\mu$, and so

$$\nabla_\mu k_\nu = \tfrac{1}{3}\mu\epsilon_{\mu\nu\rho}k^\rho \,. \tag{3.28}$$

In particular, this relation implies that k is a Killing vector. Thus, we have reduced the content of the field equation and the Bianchi identity to p being constant and (3.28). The value of the constant p is not fixed yet, since it is an overall factor in the field equation.

With these results, we can now show that there is a metric that satisfies these properties, and that it must be squashed AdS_3. It is convenient at this point to choose a coordinate system adapted to the existence of the constant-length timelike Killing vector. We may, without loss of generality, write the metric as

$$\mathrm{d}s^2 = -(\mathrm{d}t + \mathcal{A})^2 + \mathrm{d}s_2^2 \,, \tag{3.29}$$

where $\mathcal{A} = \mathcal{A}_i\,\mathrm{d}x^i$ and $\mathrm{d}s_2^2 = h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ depend only on the two spatial coordinates x^i, with $i = 1, 2$. The timelike Killing vector, which is normalised to unit length, is $k^\mu\partial_\mu = \partial/\partial t$. The associated 1-form is given by $k_\mu\,\mathrm{d}x^\mu = -(\mathrm{d}t + \mathcal{A})$. We choose a sign so that the two- and three-dimensional metrics have respective volume forms ϵ_2 and ϵ_3 related by

$$\epsilon_3 = -k \wedge \epsilon_2 \,, \tag{3.30}$$

or in components $\epsilon_{tij} = \epsilon_{ij}$. Noting that (3.23) can be written in terms of differential forms as

$$\mathrm{d}k = \tfrac{2}{3}\mu \ast k \,, \tag{3.31}$$

we see that

$$\mathcal{F} = \tfrac{2}{3}\,\mu\,\epsilon_2 \,, \tag{3.32}$$

where $\mathcal{F} := \mathrm{d}\mathcal{A}$.

The vielbein components R_{ab} of the Ricci tensor for $\mathrm{d}s^2$ are related to the vielbein components $\bar{R}_{ij}$ of the metric $\mathrm{d}s_2^2 = \mathrm{d}\bar{s}^2$ by

$$R_{00} = \tfrac{1}{4}\mathcal{F}_{ij}\mathcal{F}^{ij} \,, \qquad R_{ij} = \bar{R}_{ij} + \tfrac{1}{2}\mathcal{F}_{ik}\,\mathcal{F}_j{}^k \,, \qquad R_{0i} = \tfrac{1}{2}\bar{\nabla}^j\mathcal{F}_{ij} \,. \tag{3.33}$$

From (3.18) and (3.32) we therefore find that

$$p = \tfrac{1}{9}\mu^2 - m^2\,, \tag{3.34}$$

and

$$\bar{R}_{ij} = -\tfrac{1}{9}(\mu^2 + 27m^2)\,\delta_{ij}\,. \tag{3.35}$$

The base metric $\mathrm{d}s_2^2$ is therefore Einstein, with a negative cosmological constant, i.e. a hyperbolic space $\mathbb{H}^2$.

Locally, the base metric can be taken simply to be the standard metric on the hyperbolic plane, given by

$$\mathrm{d}s_2^2 = 9(\mu^2 + 27m^2)^{-1}\,(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,. \tag{3.36}$$

The volume form is $\epsilon_2 = 9(\mu^2 + 27m^2)^{-1}\,\sinh\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$, and so from (3.32) we may take the potential $\mathcal{A}$ to be given by

$$\mathcal{A} = 6\mu\,(\mu^2 + 27m^2)^{-1}\,\cosh\theta\,\mathrm{d}\phi\,. \tag{3.37}$$

Defining a new time coordinate $\tau = (\mu^2 + 27m^2)t/6\mu$, the metric takes the form of (3.8), which is timelike-squashed AdS_3.

We now turn to the spacelike case. A Petrov–Segre type $\mathrm{D_s}$ solution has a Ricci tensor of the form

$$R_{\mu\nu} = (p - 2m^2)g_{\mu\nu} - 3pk_\mu k_\nu\,, \tag{3.38}$$

where k is a spacelike vector field normalized so that $k^\mu k_\mu = 1$, and p is a scalar function that is not identically zero. We may perform the same analysis as in the case of type $\mathrm{D_s}$ solution; we can effectively replace $k \to \mathrm{i}k$ in the equations that result. We again derive that p must be constant and that k is a Killing vector. We may write the metric in adapted coordinates as

$$\mathrm{d}s^2 = \mathrm{d}s_2^2 + (\mathrm{d}y + \mathcal{A})^2\,, \tag{3.39}$$

where $\mathcal{A} = \mathcal{A}_i\,\mathrm{d}x^i$ and $\mathrm{d}s_2^2 = h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ depend only on the two spacetime coordinates x^i, with $i = 0, 1$. The spacelike Killing vector is $k^\mu\partial_\mu = \partial/\partial y$, and has associated 1-form $k_\mu\,\mathrm{d}x^\mu = \mathrm{d}y + \mathcal{A}$. Instead of choosing (3.30), we choose

$$\epsilon_3 = k\wedge\epsilon_2\,, \tag{3.40}$$

or in components $\epsilon_{ijy} = -\epsilon_{ij}$, and again have (3.31), (3.32), (3.34) and (3.35). The base metric is therefore anti-de Sitter spacetime AdS_2.

Locally, the base metric can be taken to be the AdS_2 metric in global coordinates,

$$\mathrm{d}s_2^2 = 9(\mu^2 + 27m^2)^{-1}(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2)\,. \tag{3.41}$$

The volume form is $\epsilon_2 = 9(\mu^2 + 27m^2)^{-1}\cosh\rho\,\mathrm{d}\tau \wedge \mathrm{d}\rho$, and so from (3.32) we may take the potential $\mathcal{A}$ to be given by

$$\mathcal{A} = 6\mu\,(\mu^2 + 27m^2)^{-1}\sinh\rho\,\mathrm{d}\tau\,. \tag{3.42}$$

Defining a new spatial coordinate $z = (\mu^2 + 27m^2)y/6\mu$, the metric takes the form of (3.13), which is spacelike-squashed AdS_3.

We have taken $\Lambda \leq 0$ for definiteness, but entirely analogous results hold for $\Lambda > 0$. The only difference is that for $\Lambda = \frac{1}{27}\mu^2$ the two-dimensional base space(time) is flat, and for $\Lambda > \frac{1}{27}\mu^2$ the base space(time) has constant positive curvature. In these cases, different choices of explicit coordinates are required.

4 AdS pp-waves

We now complete the enumeration of solutions in the literature by reviewing the AdS pp-wave solutions of TMG. Any solution of TMG that admits a null Killing vector field is (an AdS generalization of) a pp-wave [29]; we shall call such a solution an AdS pp-wave. Specifically, it was shown that in an adapted coordinate system for which the null Killing vector is $k^\mu\partial_\mu = \partial/\partial v$, the solution for generic values $\mu \neq \pm m$ can take the form[4]

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\mathrm{e}^{2m\rho}\,\mathrm{d}u\,\mathrm{d}v + [\mathrm{e}^{(m-\mu)\rho}\,f_1(u) + \mathrm{e}^{2m\rho}\,f_2(u) + f_3(u)]\,\mathrm{d}u^2\,. \tag{4.1}$$

In fact the functions $f_2(u)$ and $f_3(u)$ can be removed by means of coordinate transformations, and so the AdS pp-wave solutions are characterised by the single arbitrary function $f_1(u)$. In the special cases $\mu = \pm m$, the solutions, which can be obtained from (4.1) by taking appropriate limits, are

$$\mu = +m: \;\; \mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\mathrm{e}^{2m\rho}\,\mathrm{d}u\,\mathrm{d}v + [\rho\,f_1(u) + \mathrm{e}^{2m\rho}\,f_2(u) + f_3(u)]\,\mathrm{d}u^2\,, \tag{4.2}$$

$$\mu = -m: \;\; \mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\mathrm{e}^{2m\rho}\,\mathrm{d}u\,\mathrm{d}v + [\rho\,\mathrm{e}^{2m\rho}\,f_1(u) + \mathrm{e}^{2m\rho}\,f_2(u) + f_3(u)]\,\mathrm{d}u^2\,. \tag{4.3}$$

[4] The sign of μ can be reversed by changing the orientation of the spacetime.

Again, the functions $f_2(u)$ and $f_3(u)$ can be removed by means of coordinate transformations.

The traceless Ricci tensor takes the form

$$S_{\mu\nu} = c\, f_1(u)\, \mathrm{e}^{-(3m+\mu)\rho}\, k_\mu\, k_\nu\,, \tag{4.4}$$

where $c = \frac{1}{2}(m^2 - \mu^2)$ for $\mu \neq \pm m$ and $c = \mu$ for $\mu = \pm m$. The covariant derivative of k is

$$\nabla_\mu k_\nu = -m\epsilon_{\mu\nu\rho} k^\rho\,. \tag{4.5}$$

Combining these two expressions, using $\epsilon_{vu\rho} = +\sqrt{-g}$, provides a check that we have a solution of TMG. Examples of AdS pp-waves appearing in the literature are given in Appendix A.3.

5 Conclusion

In this paper, we have studied the algebraic classification of exact solutions in topologically massive gravity, with a cosmological constant, described by the action (1.1). We have taken a first step towards classifying all possible solutions having a specific Petrov–Segre type. Specifically, we have shown that a type D spacetime must be biaxial timelike-squashed or spacelike-squashed AdS_3.

We have found that almost all the existing solutions are locally equivalent, after coordinate transformation, to the biaxial timelike-squashed or spacelike-squashed AdS_3, or to AdS pp-waves. Thus locally, the previously known solutions with non-vanishing cosmological constant are equivalent to (3.8), (3.13) or (4.1). These are respectively of Petrov–Segre types $\mathrm{D_t}$, $\mathrm{D_s}$ and N.

In a subsequent paper [31], we shall present a large class of solutions in TMG, which belong to the Kundt class of spacetimes. These are generically of Petrov–Segre type II, but special cases are types III, N or D. They provide the first known solutions of Petrov–Segre types II and III.

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A Literature Review

Timelike- and spacelike-squashed AdS_3 and AdS pp-waves have been independently rediscovered as solutions of topologically massive gravity several times. The literature is rather fragmented, using different coordinate systems that are not obviously related. We connect these fragments here by reviewing how these three ubiquitous solutions appear in the literature.

Our aim is to be comprehensive; some of these connections have been known previously. However, we restrict ourselves to their introduction in the context of TMG. Squashed AdS_3 has been studied in other contexts; see, for example, [35, 36, 22] for references. Furthermore, we are only concerned with the local forms of the metrics here, not with their global interpretations (for example, as black hole solutions obtained by quotienting by a discrete group).

For ease of comparison, we present the literature in our $-++$ signature, and we alter the notation of the original literature so that any cosmological constant only appears via our parameter m, which has dimensions of mass, often by replacing $\ell = \pm 1/m$, where ℓ has dimensions of length. We present the metrics with the same coordinates as the original literature, although we may rearrange them. Each metric is then transformed to a canonical form, from which the conventions for $\mu^{-1}C_{\mu\nu}$ in the original literature can be traced back.

A.1 Timelike-squashed AdS_3

To discover that the solutions here are (special cases of) timelike-squashed AdS_3, we have applied the result obtained in Section 3. Namely, we found that these solutions all have Petrov–Segre type D_t, i.e. the traceless Ricci tensor is of the form $S_{\mu\nu} = \alpha(g_{\mu\nu} + 3k_\mu k_\nu)$ for some unit timelike vector k.

Recall from Section 3.2 that the metric of timelike-squashed AdS_3 can be written in the form

$$\mathrm{d}s^2 = -(\mathrm{d}t + \mathcal{A})^2 + \mathrm{d}s_2^2\,, \tag{A.1}$$

where $\mathrm{d}s_2^2$ is the metric of hyperbolic space $\mathbb{H}^2$ with squared radius $L^2 = 9/(\mu^2 + 27m^2)$, which has volume-form ϵ_2, and $\mathrm{d}\mathcal{A} = \frac{2}{3}\mu\epsilon_2$. It is helpful to review several coordinate systems. Two-dimensional hyperbolic space $\mathbb{H}^2$, with radius $L > 0$, is the upper leaf $X^0 \geq L$ of the hyperboloid

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = -L^2 \tag{A.2}$$

in the flat three-dimensional spacetime with metric

$$\mathrm{d}s^2 = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2 \,. \tag{A.3}$$

- The choice

$$X^0 = L\cosh\theta \,, \quad X^1 = L\sinh\theta\,\cos\phi \,, \quad X^2 = L\sinh\theta\,\sin\phi \tag{A.4}$$

  gives $\mathbb{H}^2$ in polar coordinates:

$$\mathrm{d}s_2^2 = L^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2) \,. \tag{A.5}$$

- The choice

$$X^0 + X^2 = \frac{L^2}{y} \,, \quad X^0 - X^2 = \frac{x^2+y^2}{y} \,, \quad X^1 = \frac{Lx}{y} \tag{A.6}$$

  gives $\mathbb{H}^2$ in Poincaré coordinates:

$$\mathrm{d}s_2^2 = \frac{L^2(\mathrm{d}x^2 + \mathrm{d}y^2)}{y^2} \,. \tag{A.7}$$

- The choice

$$X^0 = L\cosh\theta\,\cosh\phi \,, \quad X^1 = L\sinh\theta \,, \quad X^2 = L\cosh\theta\,\sinh\phi \tag{A.8}$$

  gives

$$\mathrm{d}s_2^2 = L^2(\mathrm{d}\theta^2 + \cosh^2\theta\,\mathrm{d}\phi^2) \,. \tag{A.9}$$

We hereby express the timelike-squashed AdS_3 solution of TMG in several different coordinate systems, with which we may compare the literature:

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left(-\frac{4\mu^2}{\mu^2+27m^2}(\mathrm{d}\tau + \cosh\theta\,\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2\right), \tag{A.10}$$

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left[-\frac{4\mu^2}{\mu^2+27m^2}\left(\mathrm{d}\tau + \frac{\mathrm{d}x}{y}\right)^2 + \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}\right], \tag{A.11}$$

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left(-\frac{4\mu^2}{\mu^2+27m^2}(\mathrm{d}\tau + \sinh\theta\,\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \cosh^2\theta\,\mathrm{d}\phi^2\right). \tag{A.12}$$

Relations between the coordinate systems for the $\mathbb{H}^2$ part of the metric can be obtained via the expressions for X^0, X^1 and X^2. The various τ coordinates above are distinct; relations

between them can be obtained after relating the $\mathbb{H}^2$ coordinates.

The comparison with the literature is further helped by performing an additional coordinate transformation on each of these three coordinate systems for $\mathbb{H}^2$. If we make the coordinate changes $\cosh\theta = Ar^2+B$, $\phi \to \phi/2AL^2$ in (A.5), $x = \phi/2AL^2$, $y = 1/(Ar^2+B)$ in (A.7) and $\sinh\theta = Ar^2+B$, $\phi \to \phi/2AL^2$ in (A.9), for some constants A and B, then we obtain respectively

$$\mathrm{d}s_2^2 = \frac{4L^2r^2\,\mathrm{d}r^2}{(r^2+B/A)^2-1/A^2} + \frac{(r^2+B/A)^2-1/A^2}{4L^2}\,\mathrm{d}\phi^2\,, \tag{A.13}$$

$$\mathrm{d}s_2^2 = \frac{4L^2r^2\,\mathrm{d}r^2}{(r^2+B/A)^2} + \frac{(r^2+B/A)^2}{4L^2}\,\mathrm{d}\phi^2\,, \tag{A.14}$$

$$\mathrm{d}s_2^2 = \frac{4L^2r^2\,\mathrm{d}r^2}{(r^2+B/A)^2+1/A^2} + \frac{(r^2+B/A)^2+1/A^2}{4L^2}\,\mathrm{d}\phi^2\,. \tag{A.15}$$

It follows that

$$\mathrm{d}s^2 = -(\mathrm{d}t - \tfrac{1}{3}\mu r^2\mathrm{d}\phi)^2 + \frac{r^2\,\mathrm{d}r^2}{\frac{1}{36}(\mu^2+27m^2)r^4+k_1r^2+k_0} + [\tfrac{1}{36}(\mu^2+27m^2)r^4+k_1r^2+k_0]\,\mathrm{d}\phi^2\,, \tag{A.16}$$

where k_1 and k_0 are arbitrary constants, is another way of writing the timelike-squashed AdS_3 solution. The previous three separate coordinate systems respectively correspond to the discriminant $k_1^2 - \frac{1}{9}k_0(\mu^2+27m^2)$ being positive, zero or negative.

A.1.1 With cosmological constant

Gürses: Gürses [37] considers solutions that are of Gödel type. The most general solution is in his proposition 15, which is defined by his equations (42), (43), (44), (46), and part of the unlabelled equation following (39)[5]. The solution is

$$\mathrm{d}s^2 = -[\sqrt{a_0}\,\mathrm{d}t + u_2(r,\theta)\,\mathrm{d}\theta + u_1(r,\theta)\,\mathrm{d}r]^2 + \frac{e_0^2r^2\psi(r)}{a_0}\,\mathrm{d}\theta^2 + \frac{1}{\psi(r)}\,\mathrm{d}r^2\,, \tag{A.17}$$

where

$$\psi(r) = \frac{\mu^2+27m^2}{36}r^2 + b_0 + \frac{b_1}{r^2}\,, \tag{A.18}$$

u_1 and u_2 are related by

$$\frac{\partial u_1}{\partial\theta} = \frac{\partial u_2}{\partial r} + \frac{2\mu e_0}{3\sqrt{a_0}}r\,, \tag{A.19}$$

[5] We refer to the numbering of the journal version, which differs slightly from that of the current arXiv version.

and a_0, b_0, b_1 and e_0 are arbitrary constants. The general solution of (A.19) can be expressed in terms of a potential function $U(r,\theta)$, with

$$u_1 = \frac{\partial U}{\partial r}, \quad u_2 = \frac{\partial U}{\partial \theta} - \frac{\mu e_0}{3\sqrt{a_0}} r^2 \,. \tag{A.20}$$

Making the coordinate changes $t' = \sqrt{a_0}t + U$, $\phi = e_0\theta/\sqrt{a_0}$, followed by $t' \to t$, the solution is (A.16) with $k_1 = b_0$ and $k_0 = b_1$.

An earlier work of Gürses [12] gives a solution in its equation (5). It is the above with, in the notation of (A.17), $u_1 = 0$ and $u_2 = -(c_0 + \frac{1}{3}\mu e_0 r^2)/\sqrt{a_0}$, where c_0 is a constant.

Nutku: Nutku [11] generalized the solution of Vuorio (A.25) to include a cosmological constant. Solutions that are timelike-squashed AdS_3 are given in several forms.

The solution in his equation (18), after making the coordinate relabelling $\psi = \tau$, becomes (A.12).

The solution in his equation (24), after making the coordinate relabelling $\theta = \phi$, becomes (A.16) with $k_1 = 1$ and $k_0 = 0$.

The black-hole solution in his equation (25) is

$$\begin{aligned} \mathrm{d}s^2 = & -\frac{2J - M}{6}\left(\mathrm{d}t - \frac{2\mu r^2 - 3J/\mu}{2J - M}\,\mathrm{d}\theta\right)^2 + \frac{r^2\,\mathrm{d}r^2}{\frac{1}{36}(\mu^2 + 27m^2)r^4 - \frac{1}{6}Mr^2 + \frac{1}{4}J^2/\mu^2} \\ & + \frac{6}{2J - M}[\tfrac{1}{36}(\mu^2 + 27m^2)r^4 - \tfrac{1}{6}Mr^2 + \tfrac{1}{4}J^2/\mu^2]\,\mathrm{d}\theta^2 \,. \end{aligned} \tag{A.21}$$

Making the coordinate changes

$$t' = \sqrt{\frac{2J - M}{6}}\left(t + \frac{3J\theta}{\mu(2J - M)}\right), \quad \phi = \sqrt{\frac{6}{2J - M}}\,\theta \,, \tag{A.22}$$

followed by $t' \to t$, the solution is (A.16) with $k_1 = -\frac{1}{6}M$ and $k_0 = \frac{1}{4}J^2/\mu^2$.

Clément: Clément [20] considers a Killing symmetry reduction procedure to obtain stationary rotationally symmetric solutions. The solution in his equation (18) is

$$\begin{aligned} \mathrm{d}s^2 = & -\left[\sqrt{2a}\,\mathrm{d}t + \left(\frac{3b}{\mu} - \frac{2\mu\rho}{3}\right)\frac{\mathrm{d}\theta}{\sqrt{2a}}\right]^2 + \frac{\mathrm{d}\rho^2}{\frac{1}{9}(\mu^2 + 27m^2)\rho^2 + 4(a - b)\rho + 9b^2/\mu^2} \\ & + [\tfrac{1}{9}(\mu^2 + 27m^2)\rho^2 + 4(a - b)\rho + 9b^2/\mu^2]\frac{\mathrm{d}\theta^2}{2a} \,. \end{aligned} \tag{A.23}$$

Making the coordinate changes

$$t' = \sqrt{2a}t + \frac{3b\theta}{\mu\sqrt{2a}}\,, \quad r^2 = \rho\,, \quad \phi = \sqrt{\frac{2}{a}}\theta\,, \tag{A.24}$$

followed by $t' \to t$, the solution is (A.16) with $k_1 = a - b$ and $k_0 = \frac{9}{4}b^2/\mu^2$.

Anninos–Li–Padi–Song–Strominger: Anninos, Li, Padi, Song and Strominger [22] consider "warped" AdS_3 black hole solutions of TMG. Making the coordinate relabellings $\sigma = \theta$, $u = \phi$, the solution in their equation (3.4) becomes (A.12).

A.1.2 Without cosmological constant

Vuorio: Vuorio [38] finds a stationary rotationally symmetric solution. The solution of his equation (2.21) is

$$\mathrm{d}s^2 = \frac{9}{\mu^2}[-(\mathrm{d}t + 2\,\mathrm{d}\theta - 2\cosh\sigma\,\mathrm{d}\theta)^2 + (\mathrm{d}\sigma^2 + \sinh^2\sigma\,\mathrm{d}\theta^2)]\,. \tag{A.25}$$

Making the coordinate changes $t \to 2(\tau + \phi)$, $\theta \to -\phi$, $\sigma \to \theta$, the solution is (A.10) with $m = 0$.

Percacci–Sodano–Vuorio: Percacci, Sodano and Vuorio [39] consider stationary solutions for which the timelike Killing vector has a constant scalar twist. The solution in their equation (3.20) is

$$\mathrm{d}s^2 = -3[\mathrm{d}x^2 + \tfrac{2}{3}\exp(\tfrac{1}{3}\mu x^1)\,\mathrm{d}x^0]^2 + (\mathrm{d}x^1)^2 + \tfrac{1}{3}\exp(\tfrac{2}{3}\mu x^1)\,(\mathrm{d}x^0)^2\,. \tag{A.26}$$

Making the coordinate changes $\tau = \mu x^2/(2\sqrt{3})$, $x = \mu x^0/(3\sqrt{3})$, $y = \exp(-\frac{1}{3}\mu x^1)$, the solution is (A.11) with $m = 0$.

Nutku–Baekler and Ortiz: Nutku and Baekler [17] and Ortiz [18] consider solutions formed from left-invariant 1-forms of Bianchi spaces. Some of these solutions are timelike-squashed AdS_3.

Bianchi VIII: The solution in equation (4.1) of [17] and in [18], where it is called type (a) with $a = 0$, is the triaxially squashed AdS_3

$$\begin{aligned}
\mathrm{d}s^2 &= -\lambda_0^2\sigma_3^2 + \lambda_1^2\sigma_1^2 + \lambda_2^2\sigma_2^2 \\
&= -\lambda_0^2(\mathrm{d}\psi + \sinh\theta\,\mathrm{d}\phi)^2 + \lambda_1^2(-\sin\psi\,\mathrm{d}\theta + \cos\psi\,\cosh\theta\,\mathrm{d}\phi)^2 \\
&\quad + \lambda_2^2(\cos\psi\,\mathrm{d}\theta + \sin\psi\,\cosh\theta\,\mathrm{d}\phi)^2\,,
\end{aligned} \tag{A.27}$$

where λ_0, λ_1 and λ_2 are constants that satisfy, after choosing appropriate signs, $\lambda_0+\lambda_1+\lambda_2 = 0$, and σ_i are Bianchi VIII left-invariant 1-forms that satisfy $\mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3$, $\mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1$ and $\mathrm{d}\sigma_3 = -\sigma_1 \wedge \sigma_2$. It is a solution of TMG with $\mu = \pm(\lambda_0^2 + \lambda_1^2 + \lambda_2^2)/\lambda_0\lambda_1\lambda_2$,[6] the sign depending on the orientation. If $\lambda_1 = \lambda_2$, then making the coordinate relabelling $\tau = \psi$ gives timelike-squashed AdS_3 in the form (A.12) with $m = 0$.

Bianchi III: One of the solutions, by choice of signs, in equation (4.10)[7] of [17] and in [18], where it is called type (a) with $a \neq 0$, is

$$\begin{aligned}
\mathrm{d}s^2 &= \lambda_1^2[-(2+\sqrt{3})^2\sigma_1^2 + \sigma_2^2] + \frac{36}{\mu^2}\sigma_3^2 \\
&= \lambda_1^2 \mathrm{e}^{2\theta}[-(2+\sqrt{3})^2(\cosh\theta\,\mathrm{d}x + \sinh\theta\,\mathrm{d}y)^2 + (\sinh\theta\,\mathrm{d}x + \cosh\theta\,\mathrm{d}y)^2] + \frac{36}{\mu^2}\,\mathrm{d}\theta^2 \\
&= -\frac{3+2\sqrt{3}}{6}\lambda_1^2[\sqrt{3}(\mathrm{d}x - \mathrm{d}y) + 2\mathrm{e}^{2\theta}(\mathrm{d}x + \mathrm{d}y)]^2 + \frac{36}{\mu^2}\,\mathrm{d}\theta^2 + \frac{3+2\sqrt{3}}{6}\lambda_1^2\mathrm{e}^{4\theta}(\mathrm{d}x + \mathrm{d}y)^2\,,
\end{aligned} \tag{A.28}$$

where σ_i are Bianchi III[8] left-invariant 1-forms that satisfy $\mathrm{d}\sigma_1 = \sigma_3 \wedge \sigma_2$, $\mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1$ and $\mathrm{d}\sigma_3 = 0$. Making the coordinate changes

$$\tau = \frac{\mu\sqrt{3+2\sqrt{3}}\lambda_1(x-y)}{6\sqrt{2}}\,, \quad x' = \frac{\mu\sqrt{3+2\sqrt{3}}\lambda_1(x+y)}{3\sqrt{6}}\,, \quad y' = \mathrm{e}^{-2\theta}\,, \tag{A.29}$$

followed by $x' \to x$, $y' \to y$, the solution is (A.11) with $m = 0$.

Clément Clément [40] (see also [41]) considers stationary rotationally symmetric solutions.

[6] We correct a factor of 2 in μ.
[7] We correct a typographical error in $g_{\theta\theta}$.
[8] [17] refers to these as being Bianchi VI, however they are the Bianchi III limit of Bianchi VI.

The solution in his equation (4.4) is

$$\mathrm{d}s^2 = -[\mathrm{d}t + 2c\,\mathrm{d}\theta - 2c\cosh(\tfrac{1}{3}\mu r)\,\mathrm{d}\theta]^2 + \mathrm{d}r^2 + c^2\sinh^2(\tfrac{1}{3}\mu r)\,\mathrm{d}\theta^2\,. \tag{A.30}$$

Making the coordinate changes $\tau = \frac{1}{6}\mu(t + 2c\theta)$, $\theta' = \frac{1}{3}\mu r$, $\phi = -\frac{1}{3}\mu c\theta$, followed by $\theta' \to \theta$, the solution is (A.10) with $m = 0$.

The solution in his equation (4.5) is

$$\mathrm{d}s^2 = -[\mathrm{d}t - 2c\sinh(\tfrac{1}{3}\mu r)\,\mathrm{d}\theta]^2 + \mathrm{d}r^2 + c^2\cosh^2(\tfrac{1}{3}\mu r)\,\mathrm{d}\theta^2\,. \tag{A.31}$$

Making the coordinate changes $\tau = \frac{1}{6}\mu t$, $\theta' = \frac{1}{3}\mu r$, $\phi = -\frac{1}{3}\mu c\theta$, followed by $\theta' \to \theta$, the solution is (A.12) with $m = 0$.

The solutions in his equation (4.7) have various choices of signs. Two choices of signs give

$$\mathrm{d}s^2 = -\frac{c}{d}(\mathrm{d}t \mp 2d\mathrm{e}^{\pm\mu r/3}\,\mathrm{d}\theta)^2 + \mathrm{d}r^2 + cd\mathrm{e}^{\pm 2\mu r/3}\,\mathrm{d}\theta^2\,. \tag{A.32}$$

Making the coordinate changes $\tau = \sqrt{c/d}\,t$, $x = \mp\frac{1}{3}\mu\sqrt{cd}\,\theta$, $y = \mathrm{e}^{\mp\mu r/3}$, the solution is (A.11) with $m = 0$. Another two choices of signs give

$$\mathrm{d}s^2 = -3cd\left(\mathrm{d}\theta \pm \frac{2}{3d}\mathrm{e}^{\pm\mu r/3}\,\mathrm{d}t\right)^2 + \mathrm{d}r^2 + \frac{c}{3d}\mathrm{e}^{\pm 2\mu r/3}\,\mathrm{d}t^2\,. \tag{A.33}$$

Making the coordinate changes $\tau = \sqrt{3cd}\,\theta$, $x = \pm\mu\sqrt{c/27d}\,t$, $y = \mathrm{e}^{\mp\mu r/3}$, the solution is (A.11) with $m = 0$.

A.2 Spacelike-squashed AdS_3

To discover that the solutions here are (special cases of) spacelike-squashed AdS_3, we have applied the result obtained in Section 3. Namely, we found that these solutions all have Petrov–Segre type $\mathrm{D_s}$, i.e. $S_{\mu\nu} = \alpha(g_{\mu\nu} - 3k_\mu k_\nu)$ for some unit spacelike vector k.

Recall from Section 3.2 that the metric of spacelike-squashed AdS_3 can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}s_2^2 + (\mathrm{d}y + \mathcal{A})^2\,, \tag{A.34}$$

where $\mathrm{d}s_2^2$ is the metric of AdS_2 with squared radius $L^2 = 9/(\mu^2 + 27m^2)$, which has volume form ϵ_2, and $\mathrm{d}\mathcal{A} = \frac{2}{3}\mu\epsilon_2$. It is helpful to review several coordinate systems. Two-dimensional

anti-de Sitter spacetime AdS_2, with AdS radius $L > 0$, is the hyperboloid

$$-(X^0)^2 + (X^1)^2 - (X^2)^2 = -L^2 \tag{A.35}$$

in the flat three-dimensional spacetime with metric

$$\mathrm{d}s^2 = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 - (\mathrm{d}X^2)^2 \,. \tag{A.36}$$

- The choice

$$X^0 = L\cosh\rho\,\cos\tau\,, \quad X^1 = L\sinh\rho\,, \quad X^2 = L\cosh\rho\,\sin\tau \tag{A.37}$$

gives AdS_2 in global coordinates:

$$\mathrm{d}s_2^2 = L^2(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2)\,. \tag{A.38}$$

- The choice

$$X^0 = \frac{Lt}{\rho}\,, \quad X^1 + X^2 = \frac{L^2}{\rho}, \quad X^2 - X^1 = \rho - \frac{t^2}{\rho} \tag{A.39}$$

gives AdS_2 in conformally flat coordinates that cover the Poincaré patch:

$$\mathrm{d}s_2^2 = \frac{L^2(-\mathrm{d}t^2 + \mathrm{d}\rho^2)}{\rho^2}\,. \tag{A.40}$$

- The choice

$$X^0 = L\cosh\rho\,, \quad X^1 = L\sinh\rho\,\cosh\tau\,, \quad X^2 = L\sinh\rho\,\sinh\tau \tag{A.41}$$

gives AdS_2 in coordinates that cover only $X^0 \geq L$:

$$\mathrm{d}s_2^2 = L^2(-\sinh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2)\,. \tag{A.42}$$

- The choice

$$X^0 = L\sin\tau\,, \quad X^1 = L\cos\tau\,\sinh\phi\,, \quad X^2 = L\cos\tau\,\cosh\phi \tag{A.43}$$

gives AdS_2 in coordinates that cover only $-L \leq X^0 \leq L$:

$$\mathrm{d}s_2^2 = L^2(-\mathrm{d}\tau^2 + \cos^2\tau\, \mathrm{d}\phi^2)\,. \tag{A.44}$$

We hereby express the spacelike-squashed AdS_3 solution of TMG in several different coordinate systems, with which we may compare the literature:

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left(-\cosh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \frac{4\mu^2}{\mu^2+27m^2}(\mathrm{d}z + \sinh\rho\, \mathrm{d}\tau)^2\right), \tag{A.45}$$

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left[\frac{-\mathrm{d}t^2+\mathrm{d}x^2}{x^2} + \frac{4\mu^2}{\mu^2+27m^2}\left(\mathrm{d}z + \frac{\mathrm{d}t}{x}\right)^2\right], \tag{A.46}$$

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left(-\sinh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \frac{4\mu^2}{\mu^2+27m^2}(\mathrm{d}z + \cosh\rho\, \mathrm{d}\tau)^2\right), \tag{A.47}$$

$$\mathrm{d}s^2 = \frac{9}{\mu^2+27m^2}\left(-\mathrm{d}\tau^2 + \cos^2\tau\, \mathrm{d}\phi^2 + \frac{4\mu^2}{\mu^2+27m^2}(\mathrm{d}z + \sin\tau\, \mathrm{d}\phi)^2\right). \tag{A.48}$$

Relations between the coordinate systems for the AdS_2 part of the metric can be obtained via the expressions for X^0, X^1 and X^2. The various z coordinates above are distinct; relations between them can be obtained after relating the AdS_2 coordinates.

A.2.1 With cosmological constant

Bouchareb–Clément: Bouchareb and Clément [13] construct black hole solutions. The solution in their equation (4.1) is

$$\begin{aligned}\mathrm{d}s^2 = & -\frac{\mu^2+27m^2}{3(\mu^2-9m^2)}(\rho^2-\rho_0^2)\,\mathrm{d}\varphi^2 + \frac{9}{\mu^2+27m^2}\,\frac{\mathrm{d}\rho^2}{\rho^2-\rho_0^2} \\ & + \frac{3(\mu^2-9m^2)}{4\mu^2}\left[\mathrm{d}t - \left(\frac{4\mu^2\rho}{3(\mu^2-9m^2)} + \omega\right)\mathrm{d}\varphi\right]^2.\end{aligned} \tag{A.49}$$

Making the coordinate changes

$$\tau = \frac{\rho_0(\mu^2+27m^2)\varphi}{3\sqrt{3}\sqrt{\mu^2-9m^2}}, \quad \cosh\rho' = \frac{\rho}{\rho_0}, \quad z = \frac{\sqrt{\mu^2-9m^2}(\mu^2+27m^2)(\omega\varphi - t)}{4\sqrt{3}\mu^2}, \tag{A.50}$$

and then $\rho' \to \rho$, the solution is (A.47).

Anninos–Li–Padi–Song–Strominger: Anninos, Li, Padi, Song and Strominger [22] consider "warped" AdS_3 black hole solutions of TMG. After making the coordinate relabellings $\sigma = \rho$, $u = z$, the solution in their equation (3.3) becomes (A.45).

A.2.2 Without cosmological constant

Hall–Morgan–Perjés: Hall, Morgan and Perjés [28] look for solutions of TMG with a particular (Petrov)–Segre type. The solution in their equation (61) is

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v - \tfrac{1}{9}\mu^2 v^2\,\mathrm{d}u^2 + (\mathrm{d}y - \tfrac{2}{3}\mu v\,\mathrm{d}u)^2 + 2f(u)v\,\mathrm{d}u^2\,. \tag{A.51}$$

Taking $F(u) = \int \mathrm{d}u\, f(u)$ and making the coordinate changes $u' = \int \mathrm{d}u\,\mathrm{e}^{-F(u)}$, $v' = \mathrm{e}^{F(u)}v$, we see that the function f is redundant. After taking $f = 0$, we make the coordinate changes $\hat{u} = \frac{1}{18}\mu^2 u$, $\hat{v} = 1/v + \frac{1}{18}\mu^2 u$, so the solution is

$$\mathrm{d}s^2 = -\frac{36\,\mathrm{d}\hat{u}\,\mathrm{d}\hat{v}}{\mu^2(\hat{u}-\hat{v})^2} + \left(\mathrm{d}y + \frac{12}{\mu}\frac{\mathrm{d}\hat{u}}{\hat{u}-\hat{v}}\right)^2\,. \tag{A.52}$$

Then making the coordinate changes $t = \frac{1}{2}(\hat{u}+\hat{v})$, $x = \frac{1}{2}(\hat{v}-\hat{u})$, $z = -\frac{1}{6}\mu y - \log[\frac{1}{2}(\hat{v}-\hat{u})]$, the solution is (A.46) with $m = 0$.

Nutku–Baekler and Ortiz: Nutku and Baekler [17] and Ortiz [18] consider solutions formed from left-invariant 1-forms of Bianchi spaces. Some of these solutions are spacelike-squashed AdS_3.

Bianchi VIII: The triaxially squashed AdS_3 solution of [17] and [18] is

$$\mathrm{d}s^2 = -\lambda_0^2\sigma_3^2 + \lambda_1^2\sigma_1^2 + \lambda_2^2\sigma_2^2\,, \tag{A.53}$$

where λ_0, λ_1 and λ_2 are constants that satisfy, after choosing appropriate signs, $\lambda_0+\lambda_1+\lambda_2 = 0$, and σ_i are Bianchi VIII left-invariant 1-forms that satisfy $\mathrm{d}\sigma_1 = \sigma_2\wedge\sigma_3$, $\mathrm{d}\sigma_2 = \sigma_3\wedge\sigma_1$ and $\mathrm{d}\sigma_3 = -\sigma_1\wedge\sigma_2$. It is a solution of TMG with $\mu = \pm(\lambda_0^2+\lambda_1^2+\lambda_2^2)/\lambda_0\lambda_1\lambda_2$,[9] the sign depending on the orientation.

The solution is presented in equation (4.6) of [17] as

$$\begin{aligned}\mathrm{d}s^2 = & -\lambda_0^2(\cosh\psi\,\mathrm{d}\theta + \sinh\psi\,\cos\theta\,\mathrm{d}\phi)^2 + \lambda_1^2(\sinh\psi\,\mathrm{d}\theta + \cosh\psi\,\cos\theta\,\mathrm{d}\phi)^2 \\ & + \lambda_2^2(\mathrm{d}\psi + \sin\theta\,\mathrm{d}\phi)^2\,,\end{aligned} \tag{A.54}$$

If $\lambda_0 = \lambda_1$, then making the coordinate relabellings $\tau = \theta$, $z = \psi$ gives spacelike-squashed AdS_3 in the form (A.48) with $m = 0$.

[9] We correct a factor of 2 in μ.

The solution is presented in equation (4.8) of [17] as[10]

$$\begin{aligned}\mathrm{d}s^2 &= -\lambda_0^2(-\sinh\psi\,\mathrm{d}\theta + \cosh\psi\,\cosh\theta\,\mathrm{d}\phi)^2 + \lambda_1^2(\cosh\psi\,\mathrm{d}\theta - \sinh\psi\,\cosh\theta\,\mathrm{d}\phi)^2 \\ &\quad +\lambda_2^2(\mathrm{d}\psi + \sinh\theta\,\mathrm{d}\phi)^2\,,\end{aligned} \tag{A.55}$$

If $\lambda_0 = \lambda_1$, then making the coordinate relabellings $\tau = \phi$, $\rho = \theta$, $z = \psi$ gives spacelike-squashed AdS_3 in the form (A.45) with $m = 0$.

Bianchi III: One of the solutions, by choice of signs, in equation (4.10)[11] of [17] and in [18], where it is called type (a) with $a \neq 0$, is

$$\begin{aligned}\mathrm{d}s^2 &= \lambda_1^2[-(2-\sqrt{3})^2\sigma_1^2 + \sigma_2^2] + \frac{36}{\mu^2}\sigma_3^2 \\ &= \lambda_1^2 \mathrm{e}^{2\theta}[-(2-\sqrt{3})^2(\cosh\theta\,\mathrm{d}x + \sinh\theta\,\mathrm{d}y)^2 + (\sinh\theta\,\mathrm{d}x + \cosh\theta\,\mathrm{d}y)^2] + \frac{36}{\mu^2}\,\mathrm{d}\theta^2 \\ &= -\frac{2\sqrt{3}-3}{6}\lambda_1^2\mathrm{e}^{4\theta}(\mathrm{d}x+\mathrm{d}y)^2 + \frac{36}{\mu^2}\,\mathrm{d}\theta^2 + \frac{2\sqrt{3}-3}{6}\lambda_1^2[\sqrt{3}(\mathrm{d}x-\mathrm{d}y) - 2\mathrm{e}^{2\theta}(\mathrm{d}x+\mathrm{d}y)]^2\,.\end{aligned} \tag{A.56}$$

where σ_i are Bianchi III[12] left-invariant 1-forms that satisfy $\mathrm{d}\sigma_1 = \sigma_3 \wedge \sigma_2$, $\mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1$ and $\mathrm{d}\sigma_3 = 0$. Making the coordinate changes

$$t = \frac{\mu\sqrt{2\sqrt{3}-3}\lambda_1(x+y)}{3\sqrt{6}}\,, \quad x' = \mathrm{e}^{-2\theta}\,, \quad z = \frac{\mu\sqrt{2\sqrt{3}-3}\lambda_1(x-y)}{6\sqrt{2}}\,, \tag{A.57}$$

followed by $x' \to x$, the solution is (A.46) with $m = 0$.

Ait Moussa–Clément–Leygnac: Ait Moussa, Clément and Leygnac [21] obtain black hole solutions by analytically continuing the solution of Vuorio (A.25). The solution in their equation (4) is

$$\mathrm{d}s^2 = \frac{9}{\mu^2}\left(-\frac{\rho^2-\rho_0^2}{3}\,\mathrm{d}\varphi^2 + \frac{1}{\rho^2-\rho_0^2}\,\mathrm{d}\rho^2 + 3[\mathrm{d}t - (\tfrac{2}{3}\rho + \omega)\,\mathrm{d}\varphi]^2\right)\,. \tag{A.58}$$

Making the coordinate changes $\tau = \rho_0\varphi/\sqrt{3}$, $\cosh\rho' = \rho/\rho_0$, $z = \sqrt{3}(\omega\varphi - t)/2$, and then $\rho' \to \rho$, the solution is (A.47) with $m = 0$.

[10] We correct a typographical error in the λ_1^2 term.
[11] We correct a typographical error in $g_{\theta\theta}$.
[12] [17] refers to these as being Bianchi VI, however they are the Bianchi III limit of Bianchi VI.

A.3 AdS pp-waves

To discover that the solutions here are (special cases of) the AdS pp-wave, we have applied the results of [29]. Namely, we have found that these solutions have Petrov–Segre type N, i.e. $S_{\mu\nu} = k_\mu k_\nu$ for some null vector k, and furthermore k is proportional to a Killing vector. The general pp-wave solutions with a non-vanishing cosmological constant are given by (4.1), (4.2) and (4.3); we deal later with the zero cosmological constant limit. Although the functions $f_2(u)$ and $f_3(u)$ can be removed by coordinate transformations, it is convenient to include them when making comparisons with the literature.

A.3.1 With cosmological constant

Nutku: Nutku [11] considers several solutions with a cosmological constant. The solution in his equations (16) and (17) is[13]

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\{-x\ddot{h}(u)/m - c[mx + h(u)]^{1+\mu/m}\}\,\mathrm{d}u^2}{[mx + h(u)]^2}\,, \tag{A.59}$$

where $h(u)$ is an arbitrary function and c is a constant. Making the coordinate changes $\rho = -\log(mx+h)/m$, $v' = -v + \dot{h}(mx+h)/m^2$, and then $v' \to v$, the solution is (4.1) with $f_1 = -c$, $f_2 = (2h\ddot{h} + \dot{h}^2)/m^2$, $f_3 = 0$.

Clément: Clément [20] considers a Killing symmetry reduction procedure to obtain stationary rotationally symmetric solutions. The solution in his equation (21) is[14]

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4m^2\rho^2} + 2m^2\rho\left(\mathrm{d}t^2 - \frac{\mathrm{d}\theta^2}{m^2}\right) + \frac{M}{2}(1 + c\rho^{(1-\mu/m)/2})\left(\mathrm{d}t - \frac{\mathrm{d}\theta}{m}\right)^2\,, \tag{A.60}$$

where M and c are constants. Making the coordinate changes $\rho' = (\log\rho)/2m$, $u = t - \theta/m$, $v = (t+\theta/m)/2m^2$, and then $\rho' \to \rho$, the solution is (4.1), with $f_1 = \frac{1}{2}Mc$, $f_2 = 0$, $f_3 = \frac{1}{2}M$.

Ayón-Beato–Hassaïne: Ayón-Beato and Hassaïne [16] obtain AdS pp-waves by using the general AdS pp-wave ansatz. The solutions in their equations (A3), (A5) and (A4) are

[13] We correct a typographical error in the uu component.
[14] We have absorbed a sign ambiguity by taking $m = \pm 1/l$.

respectively

$$\mathrm{d}s^2 = \frac{1}{m^2y^2}[\mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}v - (my)^{1+\mu/m}F_1(u)\,\mathrm{d}u^2]\,, \tag{A.61}$$

$$\mathrm{d}s^2 = \frac{1}{m^2y^2}[\mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}v - y^2\log(-my)F_1(u)\,\mathrm{d}u^2]\,, \tag{A.62}$$

$$\mathrm{d}s^2 = \frac{1}{m^2y^2}[\mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}v - \log(-my)F_1(u)\,\mathrm{d}u^2]\,, \tag{A.63}$$

where $F_1(u)$ is an arbitrary function, have respectively $\mu \neq \pm m$, $\mu = m$ and $\mu = -m$. Making the coordinate changes $\rho = -[\log(my)]/m$, $v \to -v$, the solutions are respectively (4.1), (4.2) and (4.3), with $f_1 = -F_1$, $f_2 = 0$, $f_3 = 0$.

They had previously found [14] AdS pp-waves by considering a correspondence between Cotton gravity with a conformally coupled scalar field and TMG [42]. The general $\mu = -m$ solution cannot be obtained through this correspondence.

Ölmez–Sarıoğlu–Tekin and Dereli–Sarıoğlu: Ölmez, Sarıoğlu and Tekin [15] consider supersymmetric solutions. The solution in their equation (9) is[15]

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\mathrm{e}^{\mp 2m\rho}\,\mathrm{d}u\,\mathrm{d}v + [\beta_2(v)\mathrm{e}^{\mp(m+\mu)\rho} + \beta_1(v)\mathrm{e}^{\mp 2m\rho} + \beta_0(v)]\,\mathrm{d}v^2\,, \tag{A.64}$$

where $\beta_0(v)$, $\beta_1(v)$ and $\beta_2(v)$ are arbitrary functions. Making the coordinate changes $u \leftrightarrow v$, $\rho \to \mp\rho$, the solution is (4.1) with $f_1 = \beta_2$, $f_2 = \beta_1$, $f_3 = \beta_0$.

Previously, Dereli and Sarıoğlu [43] considered supersymmetric solutions, but with β_0, β_1 and β_2 constants. The solution in their equations (34) to (36) is a special case of the above.

Anninos–Li–Padi–Song–Strominger: Anninos, Li, Padi, Song and Strominger [22] consider "warped" AdS$_3$ black hole solutions of TMG. The solution, for $\mu = -3m$, in their equation (3.7) is

$$\mathrm{d}s^2 = \frac{1}{m^2}\left(\frac{\mathrm{d}u^2 + \mathrm{d}x^+\,\mathrm{d}x^-}{u^2} + \frac{(\mathrm{d}x^-)^2}{u^4}\right). \tag{A.65}$$

Making the coordinate changes $\rho = -\log(mu)/m$, $u' = m^2x^-$, $v = x^+/2m^2$, and then $u' \to u$, the solution is (4.1) with $\mu = -3m$, $f_1 = 1$, $f_2 = 0$, $f_3 = 0$.

[15] We have taken $m = \mp 1/l$.

Carlip–Deser–Waldron–Wise: Carlip, Deser, Waldron and Wise [44] write down an AdS pp-wave solution. The solution, for $\mu \neq \pm m$, in their equation (21) is

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + 2\,\mathrm{d}x^+\,\mathrm{d}x^- + 2(mz)^{1+\mu/m} h(x^+)\,(\mathrm{d}x^+)^2}{m^2 z^2}\,, \tag{A.66}$$

where $h(x^+)$ is an arbitrary function. Making the coordinate changes $\rho = -\log(mz)/m$, $u = x^+$, $v = x^-$, the solution is (4.1) with $f_1 = 2h/m^2$, $f_2 = 0$, $f_3 = 0$.

Gibbons–Pope–Sezgin: Gibbons, Pope and Sezgin [29] consider supersymmetric solutions of topologically massive supergravity. They find that all such solutions are AdS pp-waves. We are using their form of the solutions above in (4.1), (4.2) and (4.3).

Garbarz–Giribet–Vásquez: Garbarz, Giribet and Vásquez [45] obtain solutions for the special values of $\mu = \pm m$.

The solution in their equation (5), for which $\mu = m$, is

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{r^2}{m^2}\left(r^2 - \frac{\kappa^2 M}{2m^2}\right)^{-2}\mathrm{d}r^2 - m^2\left(r^2 - \frac{\kappa^2 M}{2m^2}\right)\left(\mathrm{d}t^2 - \frac{\mathrm{d}\phi^2}{m^2}\right) \\ & + \left[k\log\left(\frac{r^2}{r_0^2} - \frac{\kappa^2 M}{2m^2 r_0^2}\right) + \frac{\kappa^2 M}{2}\right]\left(\mathrm{d}t - \frac{\mathrm{d}\phi}{m}\right)^2\,,\end{aligned} \tag{A.67}$$

where M, r_0, κ and k are constants. Making the coordinate changes $\mathrm{e}^{2m\rho} = m^2(r^2 - \kappa^2 M/2m^2)$, $u = t - \phi/m$, $v = -(t + \phi/m)/2$, the solution is (4.2) with $f_1 = 2km$, $f_2 = 0$, $f_3 = \kappa^2 M/2 - 2k\log(mr_0)$.

The solution in their equation (25), for which $\mu = -m$, is

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{r^2}{m^2}\left(r^2 - \frac{\kappa^2 M}{2m^2}\right)^{-2}\mathrm{d}r^2 - m^2\left(r^2 - \frac{\kappa^2 M}{2m^2}\right)\left(\mathrm{d}t^2 - \frac{\mathrm{d}\phi^2}{m^2}\right) \\ & + \left[k\left(r^2 - \frac{\kappa^2 M}{2m^2}\right)\log\left(\frac{r^2}{r_0^2} - \frac{\kappa^2 M}{2m^2 r_0^2}\right) + \frac{\kappa^2 M}{2}\right]\left(\mathrm{d}t + \frac{\mathrm{d}\phi}{m}\right)^2\,,\end{aligned} \tag{A.68}$$

again with M, r_0, κ and k constants. The same coordinate changes give (4.3) with $f_1 = 2k/m$, $f_2 = -2k\log(mr_0)/m^2$, $f_3 = \kappa^2 M/2$.

A.3.2 Without cosmological constant

To take the $m \to 0$ limit of the general $\mu \neq \pm m$ case (4.1), we rewrite the solution in the form

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\mathrm{e}^{2m\epsilon\rho}\,\mathrm{d}u\,\mathrm{d}v + \left(\mathrm{e}^{(m\epsilon-\mu)\rho} f_1(u) + \frac{(\mathrm{e}^{2m\epsilon\rho} - 1)}{2m\epsilon} f_2(u) + f_3(u)\right)\mathrm{d}u^2\,, \qquad \text{(A.69)}$$

and then take the limit $\epsilon \to 0$, resulting in

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\,\mathrm{d}u\,\mathrm{d}v + [\mathrm{e}^{-\mu\rho} f_1(u) + \rho f_2(u) + f_3(u)]\,\mathrm{d}u^2\,. \qquad \text{(A.70)}$$

The functions $f_2(u)$ and $f_3(u)$ can again be made to vanish by a coordinate transformation, but are included for comparison with the literature.

Martinez–Shepley: One of the earliest appearances of any pp-wave solution of TMG appears to be in an unpublished preprint of Martinez and Shepley [46], which has zero cosmological constant. This has been referred to in [17, 11].

Aragone: Aragone [47] considers a dreibein formalism for TMG. The solution in his equation (11) is[16]

$$\mathrm{d}s^2 = \frac{4N_0^2}{[c(u) - \mu v]^2}\,\mathrm{d}x^2 + \frac{2}{\mu}\frac{\mathrm{d}c(u)}{\mathrm{d}u}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v - 2N_0\,\mathrm{d}u\,\mathrm{d}x\,, \qquad \text{(A.71)}$$

where N_0 is a constant. Making the coordinate changes $\mathrm{e}^{\mu\rho/2} = v - c(u)/\mu$, $u' = N_0(x + v - c(u)/\mu$, $v' = -u + 4/[\mu^2 v - \mu c(u)]$, followed by $u' \to u$ and $v' \to v$, the solution is (A.70) with $f_1 = 4/\mu^2$, $f_2 = 0$, $f_3 = 0$.

Percacci–Sodano–Vuorio: Percacci, Sodano and Vuorio [39] consider stationary solutions for which the timelike Killing vector has a constant scalar twist. The solution in their equation (3.19) is

$$\mathrm{d}s^2 = \left(\frac{2}{\mu x^1}\right)^2 (\mathrm{d}x^1)^2 - \frac{1}{2}\,\mathrm{d}x^0\,\mathrm{d}x^2 - \left(\frac{\mu x^1}{8}\right)^2 (\mathrm{d}x^0)^2. \qquad \text{(A.72)}$$

In the original expression of the solution, there are functions ω_i that are not specified explicitly, but must solve a certain equation; we have chosen here $\omega_1 = 0$ and $\omega_2 = 16/\mu^3 (x^2)^2$,

[16] We correct a sign in g_{uu}.

but any other choice is equivalent by redefinition of x^0. Making the coordinate changes $u = x^0/4$, $\mathrm{e}^{-\mu\rho/2} = x^1$, $v = -x^2$, the solution is (A.70) with $f_1 = -\mu^2/4$, $f_2 = 0$, $f_3 = 0$.

Hall–Morgan–Perjés: Hall, Morgan and Perjés [28] look for solutions of TMG with a particular (Petrov)–Segre type. The solution in their equation (46) is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + 2\,\mathrm{d}x\,\mathrm{d}r - 2\mathrm{e}^{-\mu u} f(x)\,\mathrm{d}x^2\,, \tag{A.73}$$

where $f(x)$ is an arbitrary function. Making the coordinate changes $u \to \rho$, $x \to u$, $r \to v$, the solution is (A.70) with $f_1 = -2f$, $f_2 = 0$, $f_3 = 0$.

Dereli–Tucker: Dereli and Tucker [48] consider solutions with a pp-wave-like ansatz. The solution in their equations (2.14) and (2.22) is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + 2\,\mathrm{d}u\,\mathrm{d}v + 2\left[\frac{1}{\mu^2}\mathrm{e}^{\mu x} f_1(u) + \left(f_3(x) - \frac{1}{\mu} f_1(u)\right) x + f_2(u) - \frac{1}{\mu^2} f_1(u)\right] \mathrm{d}u^2\,, \tag{A.74}$$

where $f_1(u)$, $f_2(u)$ and $f_3(u)$ are arbitrary functions. Making the coordinate change $\rho = -x$, the solution is (A.70), but with the replacements in (A.70) $f_1 \to 2f_1/\mu^2$, $f_2 \to 2(f_1/\mu - f_3)$, $f_3 \to 2(f_2 - f_1/\mu^2)$.

Clément: Clément [40] considers stationary rotationally symmetric solutions. The solution in his equation (4.15) is

$$\mathrm{d}s^2 = \mathrm{d}r^2 \pm 2\sigma_0(\mathrm{d}t - \omega_0\,\mathrm{d}\theta)\,\mathrm{d}\theta - \sigma_0(c\mathrm{e}^{\mp\mu r} + br + a)(\mathrm{d}t - \omega_0\,\mathrm{d}\theta)^2\,, \tag{A.75}$$

where a, b, c ω_0 and σ_0 are constants. Making the coordinate changes $\rho = \pm r$, $u = t - \omega_0\theta$, $v = \pm\omega_0\theta$, the solution is (A.70) with $f_1 = -c\sigma_0$, $f_2 = \mp b\sigma_0$, $f_3 = -a\sigma_0$.

Deser–Steif: Deser and Steif [49] consider an impulsive pp-wave solution. The solution in their equation (7) is

$$\mathrm{d}s^2 = \mathrm{d}y^2 - \mathrm{d}u\,\mathrm{d}v + \{2\kappa^2 E[y + \mu^{-1}(\mathrm{e}^{-\mu y} - 1)]\theta(y)\delta(u) + B(u)y + C(u)\}\,\mathrm{d}u^2\,, \tag{A.76}$$

where $B(u)$ and $C(u)$ are arbitrary functions, solves $G_{\mu\nu} + \mu^{-1}C_{\mu\nu} = -\kappa^2 T_{\mu\nu}$, with $T_{uu} = E\delta(y)\delta(u)$. Considering $y > 0$, replacing $\delta(u) \to 1$, and then making the coordinate changes $\rho = y$ and $v \to -2v$, the solution is (A.70) with $f_1 = 2\kappa^2 E/\mu$, $f_2 = 2\kappa^2 E + B$, $f_3 =$

$-2\kappa^2 E/\mu + C$.

Cavaglià: Cavaglià [50] considers certain rotationally symmetric solutions. The solution in his equations (23) to (26) is

$$\mathrm{d}s^2 = -[y(u-v)]^2\,\mathrm{d}u\,\mathrm{d}v - H(u+v)\,\mathrm{d}\phi\,(\mathrm{d}u + \mathrm{d}v)\,, \tag{A.77}$$

where the function $y(u-v)$ is one of: $(\alpha/\mu)\tanh[\frac{1}{4}\alpha(u-v)-\beta]$, $-(\alpha/\mu)\tan[\frac{1}{4}\alpha(u-v)-\beta]$, or $(\alpha/\mu)/[\frac{1}{4}\alpha(u-v)-\beta]$; α and β are constants. Making the coordinate changes $\rho = \frac{1}{2}\int \mathrm{d}(u-v)y(u-v)$, $u' = \int \mathrm{d}(u+v)\,H(u+v)$, $v' = -\frac{1}{2}\phi$, and then $u' \to u$, $v' \to v$, the solution is of the form (A.70). More explicitly, we have for the respective choices of y: $\mathrm{e}^{\mu\rho/2} = \cosh[\frac{1}{4}\alpha(u-v)-\beta]$, $f_1 = -f_3 = (\alpha/2H\mu)^2$, $f_2 = 0$; $\mathrm{e}^{\mu\rho/2} = \cos[\frac{1}{4}\alpha(u-v)-\beta]$, $-f_1 = f_3 = (\alpha/2H\mu)^2$, $f_2 = 0$; $\mathrm{e}^{\mu\rho/2} = \frac{1}{4}\alpha(u-v)-\beta$, $f_1 = -(\alpha/2H\mu)^2$, $f_2 = f_3 = 0$.

Dereli–Sarıoğlu: Dereli and Sarıoğlu [43] consider supersymmetric solutions. The solution in their equations (43) to (45) is obtained as a limit of their more general solution with a non-vanishing cosmological constant discussed previously.

García–Hehl–Heinicke–Macías: García, Hehl, Heinicke and Macías [26] construct a plane-wave solution to give an example of a type N solution. The solution in their equation (124) is

$$\mathrm{d}s^2 = \mathrm{d}y^2 + \mathrm{d}x^2 - \mathrm{d}t^2 - (B\mathrm{e}^{\mu y} + Ay + C)(\mathrm{d}t - \mathrm{d}x)^2\,, \tag{A.78}$$

where A, B and C are constants. Making the coordinate changes $\rho = -y$, $u = t - x$, $v = -t - x$, the solution is (A.70) with $f_1 = -B$, $f_2 = A$, $f_3 = -C$.

Macías–Camacho: Macías and Camacho [51] consider Kerr–Schild solutions of TMG without a cosmological constant. They give two solutions explicitly.

The solution in their equation (63) is

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2[A\mu^{-1}\mathrm{e}^{\mu(\xi+Yu)} + B(\xi+Yu) + C](\mathrm{d}v + Y\,\mathrm{d}\xi + \tfrac{1}{2}Y^2\,\mathrm{d}u)^2, \tag{A.79}$$

where Y is a constant. Making the coordinate changes $\rho = -(\xi + Yu)$, $u' = v + Y\xi + \frac{1}{2}Y^2 u$, $v' = -u$, and then $u' \to u$, $v' \to v$, the solution is (A.70) with $f_1 = 2A/\mu$, $f_2 = -2B$, $f_3 = 2C$.

The solution in their equation (72) is[17]

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 - 2\,\mathrm{d}u\,\mathrm{d}v - \mu^{-1}[\mathrm{e}^{\mu\xi}\gamma(v) + \alpha\xi + C(v) + \alpha\mu^{-1}]\,\mathrm{d}v^2\,, \tag{A.80}$$

where $\gamma(v)$ and $C(v)$ are arbitrary functions, and α is a constant. Making the coordinate changes $\rho = -\xi$, $u \to -v$, $v \to u$, the solution is (A.70) with $f_1 = -\mu^{-1}\gamma(u)$, $f_2 = \mu^{-1}\alpha$, $f_3 = -\mu^{-1}[C(u) + \alpha\mu^{-1}]$.

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